



BASIC RESULTS ON INTERVAL - INCLINE MATRICES

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Abstract:

Inclines are special type of semirings and also the generalization of Boolean fuzzy, Distributive Lattice and fuzzy algebra. Here we generalized the incline as interval incline. Interval valued fuzzy is best example for interval incline. We have discussed some basic properties of interval incline and interval incline matrices.

Key Words: Incline, Interval Incline, Fuzzy

1. Introduction:

Inclines are special type of semiring, that is additively idempotent semiring which product are less than or equal to each factors Cao gave the notion of incline in 1983 and He co - authored with Kim and Roush have wrote the book incline algebra and its applications which is the first book in incline algebra [3]. Kim and Roush surveyed and outline properties of incline and incline matrices in [8]. Some basic properties of incline elements as well as regular incline elements has investigated by Meenakshi and Anbalagan in [11] and have an example which deals that incline not only contain distributive lattice which also contain non - distributive lattice A several authors studied the incline matrices like invertibility [5] indices periods [3] ect.

A number of authors [6, 7, 9, 11, 16, 17] investigated the fuzzy matrices, boolean matrices and distributive matrices. The operators for interval matrices were discussed in [14] Ganesan [2] examined the properties of interval matrices and for interval valued fuzzy sets by Turksen [19] in 1986. Shyamal and M. pal [15] studied and outlined about interval valued fuzzy matrix and then many authors studied the same in various concepts.

In this paper, we extend the concept of incline matrices to interval incline matrices as the generalization of interval valued fuzzy matrices. In section 2, we give some definitions and results needed to this paper. In section 3, we have defined interval in incline and using the set of all intervals we generalized the incline as interval incline. In section 4, we investigate the invertibility conditions and discussed the properties.

2. Preliminaries:

In this section, we gave some needed definitions and results.

Definition 2.1: ([9]). An incline is a non- empty set H with binary operations $+$, $\cdot$.

$(H \times H \rightarrow H)$ are given, denoted as addition and multiplication, such that for all $x, y, z \in H$

$$x + y = y + x, \quad x + (y + z) = (x + y) + z, \quad x(y + z) = xy + xz,$$

$$(y + z)x = yx + zx, \quad x(yz) = (xy)z, \quad x + x = x, \quad x + xy = x, \quad y + xy = y$$

Definition 2.2: ([9]). The incline H is said to be commutative, if $xy = yx$, $\forall x, y \in H$

Definition 2.3: ([9]). $(H, \leq)$ is an incline with order relation " $\leq$ " defined on H such that for $x, y \in H$, $x \leq y$ if and only if $x + y = y$. If $x \leq y$ then y is said to be dominate x .

Property 2.1: ([12]). Incline H with order relation " $\leq$ " for $x, y \in H$, $xy \leq x$ and $xy \leq y$.

Property 2.2: ([12]). Incline H with order relation " $\leq$ " for $x, y \in H$

$$x + y \geq x \text{ and } x + y \geq y.$$

Throughout this paper L^1 denotes the interval incline.

3. Basic Properties:

In this section, we define the interval incline and examine some properties and operations. To define an interval in an incline. we need the usual ordering, here we have consider ' $\leq$ ' as in definition 2.3, which implies, the properties 2.1 and 2.2 also.

Definition 3.1: In an incline L for $x_L, x_U \in L$ interval is defined by $[x_L, x_U] = \{x \in L / x_L \leq x \leq x_U\}$ and the set of all intervals denoted by L^1 . The binary operations addition and multiplication for any two elements in interval incline are given in the following has an extension of the operations on interval valued fuzzy sets [8] and incline algebra [1,9]. For any two elements

$$[x_L, x_U], [y_L, y_U] \in L^1$$

$$(i) \quad [x_L, x_U] + [y_L, y_U] = [x_L + y_L, x_U + y_U]$$

$$(ii) \quad [x_L, x_U] \cdot [y_L, y_U] = [x_L \cdot y_L, x_U \cdot y_U]$$

Proposition 3.1: The conditions in the definition 2.1, which are examine for interval in an incline.

Proof: For any three elements $[x_L, x_U], [y_L, y_U], [z_L, z_U] \in L^1$

$$\begin{aligned} (i) \quad [x_L, x_U] + [y_L, y_U] &= [x_L + y_L, x_U + y_U] \\ &= [y_L + x_L, y_U + x_U]. \\ (ii) \quad [x_L, x_U] + ([y_L, y_U] + [z_L, z_U]) &= [x_L, x_U] + [y_L + z_L, y_U + z_U] \\ &= [x_L + y_L + z_L, x_U + y_U + z_U] \\ &= ([x_L, x_U] + [y_L, y_U]) + [z_L, z_U]. \end{aligned}$$

(iii) In a similar manner of (ii) we can obtain.

$$[x_L, x_U] ([y_L, y_U][z_L, z_U]) = ([x_L, x_U][y_L, y_U])[z_L, z_U].$$

$$\begin{aligned} (iv) \quad [x_L, x_U] ([y_L, y_U] + [z_L, z_U]) &= [x_L, x_U][y_L + z_L, y_U + z_U] \\ &= [x_L(y_L + z_L), x_U(y_U + z_U)] \\ &= [x_L y_L + x_L z_L, x_U y_U + x_U z_U] \\ &= [x_L y_L, x_U y_U] + [x_L z_L, x_U z_U] \\ &= [x_L, x_U][y_L, y_U] + [x_L, x_U][z_L, z_U]. \end{aligned}$$

(v) In a similar manner of (iv), we can obtain

$$([y_L, y_U] + [z_L, z_U])[x_L, x_U] = [y_L, y_U][x_L, x_U] + [z_L, z_U][x_L, x_U].$$

$$\begin{aligned} (vi) \quad [x_L, x_U] + [x_L, x_U] &= [x_L + x_L, x_U + x_U] \\ &= [x_L, x_U]. \end{aligned}$$

$$\begin{aligned} (vii) \quad [x_L, x_U] + [x_L, x_U][y_L, y_U] &= [x_L + x_L y_L, x_U + x_U y_U] \\ &= [x_L, x_U]. \end{aligned}$$

(viii) In similar manner of (vii), we can obtain

$$[y_L, y_U] + [x_L, x_U][y_L, y_U] = [y_L, y_U].$$

From Proposition 3.1, we have set of intervals in an incline from an incline so we called which is interval incline. Throughout this paper L^1 denotes the interval incline.

Definition 3.2: An interval incline is a non-empty set of elements L^1 with binary operations addition and multiplication denoted as $+, \cdot$ defined on $L^1 \times L^1 \rightarrow L^1$.

For all $[x_L, x_U], [y_L, y_U], [z_L, z_U] \in L^1$

$$\begin{aligned} (i) \quad [x_L, x_U] + [y_L, y_U] &= [y_L, y_U] + [x_L, x_U], \\ (ii) \quad [x_L, x_U] + ([y_L, y_U] + [z_L, z_U]) &= ([x_L, x_U] + [y_L, y_U]) + [z_L, z_U], \\ (iii) \quad [x_L, x_U] ([y_L, y_U] + [z_L, z_U]) &= [x_L, x_U][y_L, y_U] + [x_L, x_U][z_L, z_U], \\ (iv) \quad ([y_L, y_U] + [z_L, z_U])[x_L, x_U] &= [y_L, y_U][x_L, x_U] + [z_L, z_U][x_L, x_U], \\ (v) \quad [x_L, x_U] ([y_L, y_U][z_L, z_U]) &= ([x_L, x_U][y_L, y_U])[z_L, z_U], \\ (vi) \quad [x_L, x_U] + [x_L, x_U] &= [x_L, x_U], \\ (vii) \quad [x_L, x_U] + [x_L, x_U][y_L, y_U] &= [x_L, x_U], \\ (viii) \quad [y_L, y_U] + [x_L, x_U][y_L, y_U] &= [y_L, y_U]. \end{aligned}$$

For example, interval values fuzzy sets, Interval distributive lattices are the best example for interval inclines. Moreover, as in ([9] page. 459), if we take any interval elements $[x_L, x_U], [y_L, y_U] \in [0, 1]$, then under the operations of γ_1, γ_2 we can obtain the interval incline L^1 .

Here we define the order relation ' $\leq$ ' for interval incline as a extension of definition 2.3.

Definition 3.3: For any two elements $[x_L, x_U], [y_L, y_U] \in L^1$

$[x_L, x_U] \leq [y_L, y_U]$ if and only if $[x_L, x_U] + [y_L, y_U] = [y_L, y_U]$.

From the definition 2.3, we have the following properties as a generalization of [12].

Property 3.1: For any two elements $[x_L, x_U], [y_L, y_U] \in L^1$,

(i) For addition

$$[x_L, x_U] + [y_L, y_U] = [x_L, x_U] + ([x_L, x_U] + [y_L, y_U]).$$

$$[x_L, x_U] + [y_L, y_U] = ([x_L, x_U] + [y_L, y_U]) + [y_L, y_U].$$

Using definition 3.2, we have the following

$$[x_L, x_U] + [y_L, y_U] \geq [x_L, x_U] \text{ and } [x_L, x_U] + [y_L, y_U] \geq [y_L, y_U].$$

(ii) For multiplication

Now, from the last two conditions in definition 3.1

$$[x_L, x_U] + ([x_L, x_U] \cdot [y_L, y_U]) = [x_L, x_U]$$

$$[y_L, y_U] + ([x_L, x_U] \cdot [y_L, y_U]) = [y_L, y_U]$$

Using definition 3.2, we have the following

$$[x_L, x_U] \geq [x_L, x_U][y_L, y_U] \text{ and } [y_L, y_U] \geq [x_L, x_U][y_L, y_U].$$

Definition 3.4: An element $[a_L, a_U]$ of L^1 is said to be idempotent, if $[a_L, a_U]^2 = [a_L, a_U]$, the set of all idempotents elements in L^1 is denoted by $I(L^1)$.

i.e., $I(L^1) = \{ [a_L, a_U] \in L^1 \mid [a_L, a_U]^2 = [a_L, a_U] \}$.

Example 3.1: Incline γ_2 will be forms Interval Incline if we take the elements all are sub - intervals of $[0, 1]$, then with the some binary operations in [10]. In this Interval Incline only two idempotent elements are $[0, 0]$ and $[1, 1]$. i.e., $I(L^1) = \{ [0, 0], [1, 1] \}$. In this interval valued Fuzzy set $I(L^1) = I \vee F S$.

Remark 3.1: In this interval incline elements having the lower limit and the upper limit are same then the interval incline coincides with the incline.

An Interval incline matrix (IIM) is a matrix with the entries are the (interval) elements belongs to the Interval incline by (L^1). In particular the interval incline matrix have the lower incline matrix and upper incline matrix, that is $A \in IIM$'s which is defined as $A = [A_L, A_U] = [a_{ijL}, a_{ijU}]$.

Let $L^1 m \times n$ ($L^1 n$) denotes the set of all $m \times n$ interval incline matrices (IIM) ($n \times n$ interval incline square matrix).

Definition 3.5: An interval incline matrices of order $m \times n$ is defined as $A = [A_L, A_U] = [a_{ijL}, a_{ijU}]$ where $a_{ij} = [a_{ijL}, a_{ijU}]$ is ij^{th} element of A interval incline matrix containing all the elements as intervals A_L is the lower matrix of A and A_U is the upper matrix of A . The following definition deals with the basic operations on IIM.

Definition 3.6:

(i) For $[A_L, A_U]$ and $[B_L, B_U] \in L^1 m \times n$, the sum is defined as

$$[A_L, A_U] + [B_L, B_U] = [A_L + B_L, A_U + B_U] = [a_{ijL} + b_{ijL}, a_{ijU} + b_{ijU}].$$

(ii) For $[A_L, A_U] = [a_{ijL}, a_{ijU}]_{m \times n}$ and $[B_L, B_U] = [b_{ijL}, b_{ijU}]_{n \times p}$,

$C = AB$ i.e., $[C_L, C_U] = [A_L, A_U][B_L, B_U]$ is defined as follows with order $m \times p$.

$$[C_L, C_U] = [A_L, A_U][B_L, B_U] = [A_L B_L, A_U B_U] = \left[\sum_k a_{ikL} b_{jkL}, \sum_k a_{ikU} b_{jkU} \right] = [c_{ikL}, c_{ikU}]$$

(iii) $A \in L^1 n$ then the transpose of A is defined as

$$A^T = [A_L^T, A_U^T] = [a_{ijL}^T, a_{ijU}^T] = [a_{jiL}, a_{jiU}].$$

(iv) For $A, B \in L^1 m n$,

$$[A_L, A_U] \leq [B_L, B_U] \text{ iff } A_L \leq B_L \text{ and } A_U \leq B_U \text{ iff } a_{ijL} \leq b_{ijL} \text{ and } a_{ijU} \leq b_{ijU}, \\ i=1, 2, \dots, m \text{ and } j=1, 2, \dots, n.$$

Remark 3.2: Like remark 3.1, each entries of A_L and A_U are same then the internal incline matrix coincides with incline matrix.

Definition 3.7: An interval Incline matrix is said to be null matrix, if all its elements are zero i.e all elements are $[0, 0]$ and the matrix is denoted by O

Definition 3.8: A matrix $P \in L^1 m \times n$ is called permutation matrix if only one entry of its every row and every column is $[1, 1]$ and other entries are $[0, 0]$.

Definition 3.9: In an Interval Incline Matrix $[A_L, A_U] = [a_{ijL}, a_{ijU}]$ of order $n \times n$ is called Identity IIM, if all the diagonal entries are $[1, 1]$ and all other entries are $[0, 0]$. Which is denoted by $[I_L, I_U]$.

That is identity matrix of IIM's special of permutation matrix.

Remark 3.3: We conclude that by the above remark 3.2, Definition 3.7 and Definition (3.9) coincides with Null matrix and Identity matrix of incline respectively. We present some basic properties of IIM's. The commutative, associative and distributive laws are valid for IIM's under the operations $(+)$ and $(\cdot)$ respectively.

Property 3.2: For any three interval incline matrices $[A_L, A_U]_{m \times n}$, $[B_L, B_U]_{n \times p}$, and $[C_L, C_U]_{m \times p}$.

$$(i) [A_L, A_U] + [B_L, B_U] = [a_{ijL} + b_{ijL}, a_{ijU} + b_{ijU}] \\ = [b_{ijL} + a_{ijL}, b_{ijU} + a_{ijU}] \\ = [B_L + A_L, B_U + A_U] \\ = [B_L, B_U] + [A_L, A_U].$$

$$(ii) [A_L, A_U] + ([B_L, B_U] + [C_L, C_U]) = ([A_L, A_U] + [B_L, B_U]) + [C_L, C_U]$$

Proof:

$$[A_L, A_U] + ([B_L, B_U] + [C_L, C_U]) = [a_{ijL}, a_{ijU}] + ([b_{ijL} + c_{ijL}, b_{ijU} + c_{ijU}]) \\ = [a_{ijL} + b_{ijL} + c_{ijL}, a_{ijU} + b_{ijU} + c_{ijU}] \\ = ([a_{ijL}, a_{ijU}] + [b_{ijL}, b_{ijU}]) + [c_{ijL}, c_{ijU}] \\ = ([A_L, A_U] + [B_L, B_U]) + [C_L, C_U].$$

$$(iii) (a) ([A_L, A_U] \cdot [B_L, B_U]) \cdot [C_L, C_U] = \sum_j \left(\sum_k [a_{ikL} \cdot b_{kjL}, a_{ikU} \cdot b_{kjU}] \right) [c_{jiL}, c_{jiU}]$$

$$= \sum_j \left[\sum_k (a_{ikL} \cdot b_{kjL}) c_{jiL}, \sum_k (a_{ikU} \cdot b_{kjU}) c_{jiU} \right] \\ = \sum_j [d_{ijL} c_{jiL}, d_{ijU} c_{jiU}]$$

Where, $d_{ijL} = \sum_k (a_{ikL} b_{kjL})$ and $d_{ijU} = \sum_k (a_{ikU} b_{kjU})$.

$$(b) [A_L, A_U] \cdot ([B_L, B_U] \cdot [C_L, C_U]) = \sum_j [a_{ijL}, a_{ijU}] \sum_k [b_{ikL} c_{kjL}, b_{ikU} c_{kjU}] \\ = \sum_j \left[\sum_k (a_{ijL} (b_{ikL} c_{kjL})), \sum_k (a_{ijU} (b_{ikU} c_{kjU})) \right].$$

From (a) and (b), we can conclude

$$([A_L, A_U] \cdot [B_L, B_U]) \cdot [C_L, C_U] = [A_L, A_U] \cdot ([B_L, B_U] \cdot [C_L, C_U]).$$

$$\begin{aligned} \text{(iv)} \quad ([A_L, A_U] + [B_L, B_U])^T &= [a_{ijL} + b_{ijL}, a_{ijU} + b_{ijU}]^T \\ &= [a_{jiL} + b_{jiL}, a_{jiU} + b_{jiU}] \\ &= [a_{jiL}, a_{jiU}] + [b_{jiL}, b_{jiU}] \\ &= [a_{ijL}, a_{ijU}]^T + [b_{ijL}, b_{ijU}]^T \\ &= [A_L, A_U]^T + [B_L, B_U]^T. \end{aligned}$$

$$\text{(v)} \quad ([A_L, A_U] \cdot [B_L, B_U])^T = [B_L, B_U]^T \cdot [A_L, A_U]^T$$

Proof:

$$\begin{aligned} ([A_L, A_U] \cdot [B_L, B_U])^T &= ([a_{ijL}, a_{ijU}] [b_{ijL}, b_{ijU}])^T \\ &= \left[\sum_k (a_{ijL} b_{ijL}), \sum_k (a_{ijU} b_{ijU}) \right]^T \\ &= \left[\sum_k (b_{kjL} a_{jkL}), \sum_k (b_{kjU} a_{jkU}) \right] \\ &= [B_L, B_U]^T [A_L, A_U]^T. \end{aligned}$$

Property 3.3: Let $[A_L, A_U] = [a_{ijL}, a_{ijU}]_{m \times n}$ be the interval incline matrix then

$$\text{(i)} \quad [A_L, A_U] + [A_L, A_U] = [A_L, A_U].$$

Proof:

$$\begin{aligned} [A_L, A_U] + [A_L, A_U] &= [a_{ijL}, a_{ijuU}] + [a_{ijL}, a_{ijuU}] \\ &= [a_{ijL} + a_{ijL}, a_{ijuU} + a_{ijuU}] \\ &= [a_{ijL}, a_{ijuU}] \\ &= [A_L, A_U]. \end{aligned}$$

$$\begin{aligned} \text{since } [A_L, A_U] + [0, 0] &= [A_L, A_U] \\ \Rightarrow [a_{ijL}, a_{ijuU}] + [0, 0] &= [a_{ijL}, a_{ijuU}]. \end{aligned}$$

Therefore, the interval incline matrix is special types of semigroup with respect to addition as well as multiplication. That is, additively idempotent, commutative over additive semigroup.

3.1 Convergence and Periodicity of Interval Incline Matrix (IIM):

Let $[A_L, A_U] = [a_{ijL}, a_{ijuU}]$ be square IIM of order n . Thus the power of $[A_L, A_U]$ is defined as $[A_L^m, A_U^m] = [A_L, A_U] [A_L, A_U] \dots [A_L, A_U]$ (m times)
 i.e., $[a_{ijL}^m, a_{ijuU}^m] = [a_{ijL}, a_{ijuU}] [a_{ijL}, a_{ijuU}] \dots [a_{ijL}, a_{ijuU}]$ (m times) and naturally $(i, j)^{th}$ element of $[A_L^m, A_U^m]$ is $[a_{ijL}^m, a_{ijuU}^m]$.

Definition 3.10: Let $[A_L, A_U] = [a_{ijL}, a_{ijuU}]$ be a square interval incline matrix of order n . If there exists an integer m such that

$$\begin{aligned} [A_L^{m+1}, A_U^{m+1}] &= [A_L^m, A_U^m] \\ \Rightarrow [a_{ijL}^{m+1}, a_{ijuU}^{m+1}] &= [a_{ijL}^m, a_{ijuU}^m] \text{ holds, then the power of an interval incline matrix is said to be} \end{aligned}$$

converge. Generally, an Interval Incline Matrix is said converge when its power converge.

Definition 3.11: Let $[A_L, A_U] = [a_{ijL}, a_{ijU}]$ be a square IIM of order n . If there exists two integers m and s such that $[a_{ijL}^{m+s}, a_{ijU}^{m+s}] = [a_{ijL}^s, a_{ijU}^s]$ holds, then m is said to be the periodicity of $[A_L, A_U] = [a_{ijL}, a_{ijU}]$ and s is the starting point of $[A_L, A_U]$ corresponding to m .

Let $n > s$ be any positive integer then $n-s > 0$ and multiplying $[A_L^{n-s}, A_U^{n-s}]$ on both sides in $[A_L^{s+m}, A_U^{s+m}] = [A_L^s, A_U^s]$, we get
 $[A_L^{n-s+s+m}, A_U^{n-s+s+m}] = [A_L^{n-s+s}, A_U^{n-s+s}]$
 $\Rightarrow [A_L^{n+m}, A_U^{n+m}] = [A_L^n, A_U^n]$,

Which means that every $n > s$ be also a starting point corresponding to m .

Remark 3.4:

Every periodicity m of an IIM $[A_L, A_U]$ is a multiple of the least periodicity d . We have generalized the following properties from [10,13,18].

Property 3.4: Let $[A_L, A_U], [B_L, B_U], [C_L, C_U] \in IIM$ with suitable order, such that $[A_L, A_U] \leq [B_L, B_U]$, implies $[A_L, A_U][C_L, C_U] \leq [B_L, B_U][C_L, C_U]$.

Proof: Let us take any three IIM's namely,

$$[A_L, A_U] = [a_{ijL}, a_{ijU}]_{m \times n}, [B_L, B_U] = [b_{ijL}, b_{ijU}]_{m \times n} \text{ and } [C_L, C_U] = [c_{ijL}, c_{ijU}]_{n \times p} \text{ such that}$$

$$[A_L, A_U] \leq [B_L, B_U].$$

$$\begin{aligned} \text{Let (I)} \quad [A_L, A_U][C_L, C_U] &= [a_{ikL}, a_{ikU}][c_{kjL}, c_{kjU}] \\ &= \sum_k [a_{ikL}c_{kjL}, a_{ikU}c_{kjU}] \\ &= [d_{ijL}, d_{ijU}]_{m \times p}, \end{aligned}$$

$$\text{Where } d_{ijL} = \sum_k a_{ikL}c_{kjL} \text{ and } d_{ijU} = \sum_k a_{ikU}c_{kjU} \text{ and}$$

$$\begin{aligned} \text{(II)} \quad [B_L, B_U][C_L, C_U] &= [b_{ikL}, b_{ikU}][c_{kjL}, c_{kjU}] = \sum_k [b_{ikL}c_{kjL}, b_{ikU}c_{kjU}] \\ &= [e_{ijL}, e_{ijU}]_{m \times p} \end{aligned}$$

$$\text{Where } e_{ijL} = \sum_k b_{ikL}c_{kjL} \text{ and } e_{ijU} = \sum_k b_{ikU}c_{kjU}$$

$$\text{Since, } [A_L, A_U] \leq [B_L, B_U] \Rightarrow a_{ijL} \leq b_{ijL} \text{ and } a_{ijU} \leq b_{ijU}$$

$$\text{Again, } [C_L, C_U] = [c_{ijL}, c_{ijU}], \text{ the elements of } c_{ijL} \text{ and } c_{ijU} \in L^1 \text{ } n \times p$$

$$d_{ijL} = \sum_k a_{ikL}c_{kjL} \leq \sum_k b_{ikL}c_{kjL} = e_{ijL} \text{ and } d_{ijU} = \sum_k a_{ikU}c_{kjU} \leq \sum_k b_{ikU}c_{kjU} = e_{ijU}$$

$$\text{Hence, } [A_L, A_U][C_L, C_U] \leq [B_L, B_U][C_L, C_U].$$

Property 3.5: For $A \in L^1 n$ if $[A_L^p, A_U^p] \leq [A_L^q, A_U^q]$ or $[A_L^p, A_U^p] \geq [A_L^q, A_U^q]$ holds for $p \leq q$, then $[A_L, A_U]$ converges.

Proof: If $[A_L^p, A_U^p] \leq [A_L^q, A_U^q] \Rightarrow [a_{ijL}^p, a_{ijU}^p] \leq [a_{ijL}^q, a_{ijU}^q]$ holds then

$$a_{ijL}^{(p)} \leq a_{ijL}^{(q)} \text{ and } a_{ijU}^{(p)} \leq a_{ijU}^{(q)} \text{ we have}$$

$$a_{ijL}^{(p)} \leq a_{ijL}^{(q)} \leq a_{ijL}^{(q+1)} \leq \dots \text{ and } a_{ijU}^{(p)} \leq a_{ijU}^{(q)} \leq a_{ijU}^{(q+1)} \leq \dots$$

Since, $a_{ijL}^{(p)} \leq a_{ijL}^{(q)}$, $a_{ijU}^{(p)} \leq a_{ijU}^{(q)}$, therefore a finite number of distinct lower IIM and upper IIM in the corresponding interval (position) occur in the power of $[A_L, A_U] = A$. So that,

$$a_{ijL}^{(p)} \leq a_{ijL}^{(q)} \leq a_{ijL}^{(q+1)} \leq \dots \leq a_{ijL}^{(s)} = a_{ijL}^{(s+1)} = \dots \text{ and}$$

$$a_{ijU}^{(p)} \leq a_{ijU}^{(q)} \leq a_{ijU}^{(q+1)} \leq \dots \leq a_{ijU}^{(t)} = a_{ijU}^{(t+1)} = \dots \text{ for some finite } s \text{ and } t, \text{ where } s \geq t \text{ or } s \leq t.$$

Simply a finite number of distinct interval incline matrices occur in the power of $A = [A_L, A_U]$.

Hence, $[A_L, A_U]$ converges.

Similarly, it can be shown when $[A_L^p, A_U^p] \geq [A_L^q, A_U^q]$.

Property 3.6: $[A_L, A_U]$ be $LIn \times n$ for all $i, j \leq n$ if there exists $k \leq n$ such that $a_{ijL} \leq a_{ikL}a_{kjL}$ and $a_{ijU} \leq a_{ikU}a_{kjU}$ then $[A_L, A_U]$ converge to $[A_L^p, A_U^p]$, for some $p \leq n-1$.

4. Invertible Interval Incline Matrices:

In this section we have investigated the invertible properties of IIM's as the extension of [3,4]. For any positive integer n , always stands for any one of

$\{1, 2, 3, \dots\}$. $[n]$ denotes the least common multiple of the integers $1, 2, \dots, n$.

The set $M(L^1n)$ constitute a partially ordered monoid with respect to the matrix multiplication.

Definition 4.1: Let S be a semigroup and $[a_L, a_U] \in S$. Assume that $[a_L^k, a_U^k] = [a_L^{k+d}, a_U^{k+d}]$ for some positive integers k and d are called an index and period $[a_L, a_U]$ respectively. In this case, we say that $a = [a_L, a_U]$ has index.

Definition 4.2: An interval incline L^1 with $[0, 0]$ and $[1, 1]$ are called an integral interval incline if it has no non-zero elements $[x_L, x_U]$ and $[y_L, y_U]$ such that $[x_L y_L, x_U y_U] = [0, 0]$ and $[x_L, x_U] + [y_L, y_U] = [1, 1]$.

Definition 4.3: An interval incline matrix $[A_L, A_U] \in M(L^1n)$ is said to be right invertible (left invertible), if $[A_L B_L, A_U B_U] = [I, I]$, $([B_L A_L, B_U A_U] = [I, I])$ for some $[B_L, B_U] \in M(L^1n)$. The matrix $[B_L, B_U]$ is called a right inverse (left inverse) of $[A_L, A_U]$.

If $[A_L, A_U]$ is both right and left invertible then it is said to be invertible and $[B_L, B_U]$, denoted by $[A_L^{-1}, A_U^{-1}]$ is called an inverse of it.

Remark 4.1: A matrix of IIM's is invertible, then its right and left inverse coincide with the inverse it.

Lemma 4.1: $I(L^1)$ is interval distributive lattice.

Proof: Since $[0, 0], [1, 1] \in I(L^1)$ we see that $I(L^1)$ is non-empty.

$$\begin{aligned} \text{For any } [x_L, x_U], [y_L, y_U] \in I(L^1), \text{ we have } ([x_L, x_U] + [y_L, y_U])^2 &= ([x_L + y_L, x_U + y_U])^2 \\ &= [(x_L + y_L)^2, (x_U + y_U)^2] \\ &= [x_L^2 + 2x_L y_L + y_L^2, x_U^2 + 2x_U y_U + y_U^2] \\ &= [x_L + x_L y_L + x_L y_L + y_L, x_U + x_U y_U + x_U y_U + y_U] \end{aligned}$$

Suppose that $x_L^2 = x_L$, $x_U^2 = x_U$ and $y_L^2 = y_L$, $y_U^2 = y_U$

$$\begin{aligned} &= [(x_L + x_L y_L) + (x_L y_L + y_L), (x_U + x_U y_U) + (x_U y_U + y_U)] \\ &= [x_L + y_L, x_U + y_U] \quad (\because \text{By Definition(3.1)}) \\ &= [x_L, x_U] + [y_L, y_U] \end{aligned}$$

$$\begin{aligned} ([x_L, x_U][y_L, y_U])^2 &= [x_L y_L, x_U y_U]^2 \\ &= [x_L y_L, x_U y_U][x_L y_L, x_U y_U] \end{aligned}$$

$$\begin{aligned} &= [x_L^2 y_L^2, x_U^2 y_U^2] \\ &= [x_L y_L, x_U y_U] \\ &= [x_L, x_U] [y_L, y_U] \end{aligned}$$

Hence, $[x_L, x_U] + [y_L, y_U] \in I(L^1)$, $[x_L y_L, x_U y_U] \in I(L^1)$ and

$$[x_L, x_U] + [y_L, y_U] = [x_L + y_L, x_U + y_U] = \vee \{ [x_L, x_U], [y_L, y_U] \} \in I(L^1).$$

If $[x_L, x_U] \leq [z_L, z_U]$ and $[y_L, y_U] \geq [z_L, z_U]$ for some $[z_L, z_U] \in I(L^1)$, then

$$[x_L y_L, x_U y_U] \geq [z_L^2, z_U^2] = [z_L, z_U], \text{ which means that}$$

$$[x_L y_L, x_U y_U] = \wedge \{ [x_L, x_U], [y_L, y_U] \} \in I(L^1)$$

Thus, $I(L^1)$ is interval distributive lattice.

Lemma 4.2: If $[A_L, A_U] \in M(L^1, n)$ then $[A_L^{n^2[n]}, A_U^{n^2[n]}] \leq [A_L^{n^2[n]-[n]}, A_U^{n^2[n]-[n]}]$

Proof:

Let $A = [A_L, A_U] = [a_{ijL}, a_{ijU}]$ and $k = n^2[n]$. Denote $[A_L^p, A_U^p] = [a_{ijL}^{(p)}, a_{ijU}^{(p)}]$.

Then for any $i, j \in \underline{n}$, we have $[a_{ijL}^{(k)}, a_{ijU}^{(k)}] = \sum_{i_1, i_2, \dots, i_{k-1} \in \underline{n}} [a_{ii_1L}, a_{ii_1U}] [a_{i_1i_2L}, a_{i_1i_2U}] \dots [a_{i_{k-1}jL}, a_{i_{k-1}jU}]$

Consider any term $[a_{ii_1L}, a_{ii_1U}] [a_{i_1i_2L}, a_{i_1i_2U}] \dots [a_{i_{k-1}jL}, a_{i_{k-1}jU}]$ of $[a_{ijL}^{(k)}, a_{ijU}^{(k)}]$

Divide it into $n[n]$ products of u elements in order,

$$[a_{ii_1L}, a_{ii_1U}] [a_{i_1i_2L}, a_{i_1i_2U}] \dots [a_{i_{n-1}i_nL}, a_{i_{n-1}i_nU}] [a_{i_ni_{n+1}L}, a_{i_ni_{n+1}U}] \dots [a_{i_{k-n}i_{k-n+1}L}, a_{i_{k-n}i_{k-n+1}U}] \dots [a_{i_{k-1}jL}, a_{i_{k-1}jU}]$$

For every product of $\underline{n}$ interval elements, the number of the subscripts is equal to $n+1$ their values are in $\underline{n}$. Hence there exists two subscripts such that their value are same.

Therefore, every product of n interval elements contains a sub product of type $[a_{pp_1L}, a_{pp_1U}] \dots [a_{p_r p_r L}, a_{p_r p_r U}]$ from

the every product of n interval elements above, select exactly one subproduct of such type. Then there exists a integer $r \in \underline{n}$ such that $[n]$ subproduct of r elements are chosen atleast. In fact or else for every $r \in \underline{n}$, the number of selected subproduct of r elements is smaller than $[n]$, and so the total number of the selected subproduct is smaller than $n[n]$.

Now, select exactly $([n] / r)$ subproduct of r interval elements and leave them from

$[a_{ii_1L}, a_{ii_1U}] [a_{i_1i_2L}, a_{i_1i_2U}] \dots [a_{i_{k-1}jL}, a_{i_{k-1}jU}]$. Then we obtain a term of $[a_{ijL}^{k-[n]}, a_{ijU}^{k-[n]}]$. Hence we have,

$$[a_{ii_1L}, a_{ii_1U}] [a_{i_1i_2L}, a_{i_1i_2U}] \dots [a_{i_{k-1}jL}, a_{i_{k-1}jU}] \leq [a_{ijL}^{k-[n]}, a_{ijU}^{k-[n]}]$$

Since, the inequality above holds for any term

$$[a_{ii_1L}, a_{ii_1U}] [a_{i_1i_2L}, a_{i_1i_2U}] \dots [a_{i_{k-1}jL}, a_{i_{k-1}jU}] \text{ of } [a_{ijL}^k, a_{ijU}^k],$$

$$\text{It follows that } [a_{ijL}^k, a_{ijU}^k] \leq [a_{ijL}^{k-[n]}, a_{ijU}^{k-[n]}].$$

This completes the proof.

Lemma 4.3: If D is a interval distributive lattice and $A = [A_L, A_U] \in D(L^1, n)$, where $D(L^1, n)$ - matrices over interval distributive lattice of order n , then $[A_L^n, A_U^n] \leq [A_L^{n+[n]}, A_U^{n+[n]}]$.

Proof: Let $A = [A_L, A_U] = [a_{ijL}, a_{ijU}]$ then for any $i, j \in \underline{n}$ we have that

$$\begin{aligned} & [a_{i_1 L}, a_{i_1 U}] [a_{i_2 L}, a_{i_2 U}] \cdots [a_{i_{n-1} L}, a_{i_{n-1} U}] [a_{ijL}^{(n)}, a_{ijU}^{(n)}] \\ &= \sum_{i_1, i_2, \dots, i_{n-1} \in \underline{n}} [a_{i_1 L}, a_{i_1 U}] [a_{i_2 L}, a_{i_2 U}] \cdots [a_{i_{n-1} L}, a_{i_{n-1} U}]. \end{aligned}$$

Consider any term, $[a_{i_1 L}, a_{i_1 U}] [a_{i_2 L}, a_{i_2 U}] \cdots [a_{i_{n-1} L}, a_{i_{n-1} U}]$ of $[a_{ijL}^{(n)}, a_{ijU}^{(n)}]$.

Put $i_0 = i$ and $i_n = j$, since $i_u \in \underline{n}$ for all u ($0 \leq u \leq n$) there exists p and t such that $0 \leq p < t \leq n$ and $i_p = i$. Then we obtain that,

$$\begin{aligned} & [a_{i_0 i_1 L}, a_{i_0 i_1 U}] [a_{i_1 i_2 L}, a_{i_1 i_2 U}] \cdots [a_{i_{n-1} i_n L}, a_{i_{n-1} i_n U}] \\ &= [a_{i_0 i_1 L}, a_{i_0 i_1 U}] \cdots [a_{i_{p-1} i_p L}, a_{i_{p-1} i_p U}] [b_L, b_U] [a_{i_t i_{t+1} L}, a_{i_t i_{t+1} U}] \cdots [a_{i_{n-1} i_n L}, a_{i_{n-1} i_n U}] \\ &= [a_{i_0 i_1 L}, a_{i_0 i_1 U}] \cdots [a_{i_{p-1} i_p L}, a_{i_{p-1} i_p U}] [b_L, b_U] \cdots [a_{i_t i_{t+1} L}, a_{i_t i_{t+1} U}] \cdots [a_{i_{n-1} i_n L}, a_{i_{n-1} i_n U}] \\ &\leq [a_{i_0 i_n L}^{(n+[n])}, a_{i_0 i_n U}^{(n+[n])}]. \end{aligned}$$

Where $[b_L, b_U] = [a_{i_p i_{p+1} L}, a_{i_p i_{p+1} U}] \cdots [a_{i_{t-1} i_t L}, a_{i_{t-1} i_t U}]$ and $l = \frac{[n]}{(t-p)+1}$.

Since the inequality above holds for any term of $[a_{ijL}^{(n)}, a_{ijU}^{(n)}]$, we see that

$$[a_{ijL}^{(n)}, a_{ijU}^{(n)}] \leq [a_{ijL}^{(n+[n])}, a_{ijU}^{(n+[n])}].$$

This completes the proof.

Lemma 4.4: If D is an interval distributive lattice and $A = [A_L, A_U] \in D(L^n)$ then

$$[A_L^{n^2[n]}, A_U^{n^2[n]}] = [A_L^{n^2[n]+[n]}, A_U^{n^2[n]+[n]}].$$

Proof: By Lemma 4.2, we have that

$$\begin{aligned} [A_L^{n^2[n]+[n]}, A_U^{n^2[n]+[n]}] &= [A_L^{n^2[n]} A_L^{[n]}, A_U^{n^2[n]} A_U^{[n]}] \\ &\leq [A_L^{n^2[n]-[n]} A_L^{[n]}, A_U^{n^2[n]-[n]} A_U^{[n]}] = [A_L^{n^2[n]}, A_U^{n^2[n]}] \end{aligned}$$

By Lemma 4.3, we obtain that

$$\begin{aligned} [A_L^{n^2[n]+[n]}, A_U^{n^2[n]+[n]}] &= [A_L^{n+[n]} A_L^{n^2[n]-n}, A_U^{n+[n]} A_U^{n^2[n]-n}] \\ &\geq [A_L^n A_L^{n^2[n]-n}, A_U^n A_U^{n^2[n]-n}] = [A_L^{n^2[n]}, A_U^{n^2[n]}]. \end{aligned}$$

This completes the proof.

4.1 Some Conditions for Invertibility of IIM's:

In this section, we take the interval incline L^1 with $[0, 0]$ and $[1, 1]$ where $[1, 1]$ is the multiplicative identity. Here we give some necessary and sufficient conditions for the IIM's to be invertible [4].

Theorem 4.1: If $A = [A_L, A_U] \in M(L^n)$ is right invertible, then $A^{[n]} = [A_L^{[n]}, A_U^{[n]}] = [I_L, I_U] = I$.

Proof: Since $A = [A_L, A_U]$ is right invertible,

$$[A_L X_L, A_U X_U] = [I_L, I_U] \text{ for some } [X_L, X_U] \in M(L^n).$$

Let $k = n^2[n]$ and $d = [n]$ then $[A_L^{k+d}, A_U^{k+d}] \leq [A_L^k, A_U^k]$ by Lemma 4.8 we have that

$$\begin{aligned} A^d &= [A_L^d, A_U^d] [I_L, I_U] \\ &= [A_L^d, A_U^d] ([A_L, A_U] \cdots ([A_L, A_U] ([A_L X_L, A_U X_U]) [X_L, X_U]) \cdots [X_L, X_U]) \\ [A_L^d, A_U^d] [A_L^k, A_U^k] [X_L^k, X_U^k] &= [A_L^{k+d} X_L^k, A_U^{k+d} X_U^k] \leq [A_L^k X_L^k, A_U^k X_U^k] \end{aligned}$$

$$= [A_L, A_U] \dots [A_L X_L, A_U X_U] \dots [X_L, X_U] = [I_L, I_U] = I.$$

$$\text{i.e., } [A_L^d, A_U^d] \leq [I_L, I_U].$$

$$\text{Now let } A = [A_L, A_U] = [a_{ijL}, a_{ijU}] \text{ and } X = [X_L, X_U] = [x_{ijL}, x_{ijU}].$$

$$\text{From } [A_L^d X_L^d, A_U^d X_U^d] = [I_L, I_U] \text{ it follows that}$$

$$\sum_{s \in \underline{n}} [a_{isL}^{(d)} x_{siL}^{(d)}, a_{isU}^{(d)} x_{siU}^{(d)}] = [1, 1].$$

$$\text{Since, } [A_L^d, A_U^d] \leq [I_L, I_U], \text{ we have that } [a_{ijL}^{(d)}, a_{ijU}^{(d)}] = [0, 0] \text{ for } i \neq j.$$

$$\text{Therefore, for every } i \in \underline{n} \text{ we obtain that } a_{ii}^{(d)} = [a_{iiL}^{(d)}, a_{iiU}^{(d)}]$$

$$= \sum_{s \in \underline{n}} [a_{isL}^{(d)}, a_{isU}^{(d)}] \geq \sum_{s \in \underline{n}} [a_{isL}^{(d)} x_{siL}^{(d)}, a_{isU}^{(d)} x_{siU}^{(d)}] = [1, 1].$$

$$\text{i.e., } [a_{iiL}^{(d)}, a_{iiU}^{(d)}] = [1, 1].$$

$$\text{Consequently, } [A_L^d, A_U^d] = [I_L, I_U].$$

$$\text{Therefore, } A_L^d = A_U^d \Rightarrow A^d = I..$$

Theorem 4.2: Let $A = [A_L, A_U] \in M(L^1 n)$ be a permutation matrix and an invertible matrix then

$$[A_L^T A_L, A_U^T A_U] = [A_L A_L^T, A_U A_U^T] = [I_L, I_U]$$

Proof: Since A is invertible,

$$\text{i.e., } [A_L, A_U][X_L, X_U] = [X_L, X_U][A_L, A_U] = [I, I] \text{ for some } [X_L, X_U] \in M(L^1 n).$$

$$\text{Put } A = a_{ij} = [a_{ijL}, a_{ijU}], X = x_{ij} = [x_{ijL}, x_{ijU}] \text{ and } A = [A_L, A_U] \text{ is invertible.}$$

$$\text{That is, } [A_L, A_U][X_L, X_U] = [I, I].$$

$$\text{We have that if } i \neq j, \text{ then } \sum_{s \in \underline{n}} [a_{isL} x_{sjL}, a_{isU} x_{sjU}] = [0, 0]$$

$$\text{So } [a_{isL} x_{sjL}, a_{isU} x_{sjU}] = [0, 0] \text{ for all } s \in \underline{n}.$$

$$\text{By } [X_L A_L, X_U A_U] = [I_L, I_U] \text{ we obtain that } \sum_{s \in \underline{n}} [a_{siL}, a_{siU}] \geq \sum_{s \in \underline{n}} [x_{isL} a_{siL}, x_{isU} a_{siU}] = [1, 1] = 1.$$

$$\text{and } \sum_{s \in \underline{n}} [x_{isL}, x_{isU}] \geq \sum_{s \in \underline{n}} [x_{isL} a_{siL}, x_{isU} a_{siU}] = [1, 1] \text{ for all } i \in \underline{n}.$$

Hence we see that

$$\begin{aligned} [a_{ijL}, a_{ijU}] &= [a_{ijL}, a_{ijU}][1, 1] = [a_{ijL}, a_{ijU}] \left(\sum_{s \in \underline{n}} [x_{jsL}, x_{jsU}] \right) = \sum_{s \in \underline{n}} [a_{ijL} x_{jsL}, a_{ijU} x_{jsU}] \\ &= [a_{ijL} x_{jiL}, a_{ijU} x_{jiU}] + \sum_{s \neq i} [a_{ijL} x_{jsL}, a_{ijU} x_{jsU}] = [a_{ijL} x_{jiL}, a_{ijU} x_{jiU}] + [0, 0] \\ &= [a_{ijL} x_{jiL}, a_{ijU} x_{jiU}] + \sum_{s \neq i} [a_{sjL} x_{jiL}, a_{sjU} x_{jiU}] = \sum_{s \in \underline{n}} [a_{sjL} x_{jiL}, a_{sjU} x_{jiU}] \\ &= \left(\sum_{s \in \underline{n}} [a_{sjL}, a_{sjU}] \right) [x_{jiL}, x_{jiU}] = [1, 1] [x_{jiL}, x_{jiU}] = [x_{jiL}, x_{jiU}] \text{ for all } i, j \in \underline{n}. \end{aligned}$$

$$\text{Therefore, } X = A^T \text{ or } [X_L, X_U] = [A_L^T, A_U^T].$$

Theorem 4.3: If $A = [A_L, A_U] \in M(L^1 n)$ and f_A is surjective mapping then A is right invertible.

Proof: Since f_A is surjective mapping, there are some column vectors

$$(x_{1i}, x_{2i}, \dots, x_{ni})^T \in (L^1 n) \quad (i \in \underline{n}) \text{ such that}$$

$$[A_L, A_U] \begin{bmatrix} [x_{11L}, x_{11U}] \\ [x_{21L}, x_{21U}] \\ \dots \\ [x_{n1L}, x_{n1U}] \end{bmatrix} = \begin{bmatrix} [1,1] \\ [0,0] \\ \dots \\ [0,0] \end{bmatrix} \cdot [A_L, A_U] \begin{bmatrix} [x_{12L}, x_{12U}] \\ [x_{22L}, x_{22U}] \\ \dots \\ [x_{n2L}, x_{n2U}] \end{bmatrix} = \begin{bmatrix} [0,0] \\ [1,1] \\ \dots \\ [0,0] \end{bmatrix} \dots [A_L, A_U] \begin{bmatrix} [x_{1nL}, x_{1nU}] \\ [x_{2nL}, x_{2nU}] \\ \dots \\ [x_{nnL}, x_{nnU}] \end{bmatrix} = \begin{bmatrix} [0,0] \\ [0,0] \\ \dots \\ [1,1] \end{bmatrix}.$$

Put $X = [X_L, X_U] = [x_{ijL}, x_{ijU}]$ then $[X_L, X_U] \in M(L^1 n)$ and $[A_L, A_U][X_L, X_U] = [I, I]$

i.e., $A = [A_L, A_U]$ is right invertible.

Theorem 4.4: If $A = [A_L, A_U] \in M(L^1 n)$, then the following statements are equivalent to each other.

- A is right invertible
- A is left invertible
- A is invertible
- $AA^T = I$
- $A^T A = I$
- $AA^T = A^T A = I$
- $A^k = I$ for some positive integer k
- f_A is a surjective mapping
- f_A is bijective mapping
- f_A preserves the scalar product
- $A \in M(L^1 n)$ and f_A is an injective mapping
- A has index and f_A is an injective mapping

Proof: $(b) \Rightarrow (c)$ it is proved by the same method as the proof of Theorem 4.1

The equivalence of the first seven statements is proved in the procedures of

$$(a) \Rightarrow (g) \Rightarrow (c) \Rightarrow (f) \Rightarrow (d) \Rightarrow (a),$$

$$(b) \Rightarrow (g) \Rightarrow (c) \Rightarrow (f) \Rightarrow (e) \Rightarrow (b).$$

By using the theorem 4.1 and Theorem 4.2

$$(i) \Rightarrow (h), \text{ which is obvious,}$$

$$(h) \Rightarrow (a) \text{ which is Theorem 4.3,}$$

$$(c) \Rightarrow (i) \text{ which is Theorem 4.2,}$$

$$[A_L^T A_L, A_U^T A_U] = [A_L A_L^T, A_U A_U^T] = [I, I].$$

If $f_A(x) = f_A(y)$ then $[A_L x, A_U x] = [A_L y, A_U y]$ and so

$$\begin{aligned} [x_L, x_U] &= [A_L^T A_L, A_U^T A_U][x_L, x_U] = [A_L^T, A_U^T][A_L, A_U][x_L, x_U] \\ &= [A_L^T, A_U^T][A_L, A_U]y = [A_L^T A_L, A_U^T A_U][y_L, y_U]. \end{aligned}$$

Hence f_A is an injective mapping.

For any $[y_L, y_U] \in (L^1 n)$, put $[x_L, x_U] = [A_L^T y, A_U^T y]$, then $[x_L, x_U] \in (L^1 n)$ and

$$f_A(x) = [A_L, A_U][A_L^T y_L, A_U^T y_U] = [A_L A_L^T, A_U A_U^T][y_L, y_U] = I[y_L, y_U] = [y_L, y_U].$$

Hence f_A is an surjective mapping.

$$(j) \Rightarrow (e) \text{ suppose } (j) \text{ holds. Then for any } [x_L, x_U], [y_L, y_U] \in (L^1 n),$$

$$\begin{aligned} [x_L^T, x_U^T][A_L^T, A_U^T][A_L y_L, A_U y_U] &= ([A_L x_L, A_U x_U])^T [A_L y_L, A_U y_U] \\ &= f_A(x)^T f_A(y) = [x_L, x_U]^T [y_L, y_U] = [x_L^T y_L, x_U^T y_U]. \end{aligned}$$

Let $A = [A_L, A_U] = [A_{sL}, A_{sU}]$. For any $i \in \underline{n}$, let $[e_{iL}, e_{iU}]$ be the i^{th} unit vector in $(L^1 n)$,

i.e., the only entry ' i '. If it is $[1,1]$ and the other entries are $[0,0]$, for any $i \in \underline{n}$ replace $[x_L, x_U]$ and $[y_L, y_U]$ with $[e_{iL}, e_{iU}]$ respectively in the expression above. Then we have that $\sum_{s \in \underline{n}} [a_{siL} a_{sjL}, a_{siU} a_{sjU}] = \begin{cases} [1,1], & i = j \\ 0, & i \neq j \end{cases}$.

Hence $[A_L^T A_L, A_U^T A_U] = [I, I]$.

(e) $\Rightarrow$ (j) If $[A_L^T A_L, A_U^T A_U] = [I, I]$, then for any $[x_L, x_U], [y_L, y_U] \in (L^1 n)$, we obtain that

$$f_A^T([x_L, x_U]) f_A([y_L, y_U]) = (A[x_L, x_U])^T (A[y_L, y_U]) = [x_L, x_U]^T A^T A [y_L, y_U] \\ = [x_L, x_U]^T I [y_L, y_U] = [x_L, x_U]^T [y_L, y_U].$$

Hence (j) holds.

(c) $\Rightarrow$ (k), Let $A = [A_L, A_U] = [a_{ijL}, a_{ijU}] \in M(L^1 n)$ be an invertible matrix. It follows from

$$(d) \text{ that for any } i \in \underline{n}, \sum_{s \in \underline{n}} [a_{isL}, a_{isU}] \geq \sum_{s \in \underline{n}} [a_{isL}^2, a_{isU}^2] = [1,1]$$

i.e., $\sum_{s \in \underline{n}} [a_{isL}, a_{isU}] = [1,1]$. Hence for any $i, j \in \underline{n}$, we have that

$$[a_{ijL}, a_{ijU}] = [a_{ijL}, a_{ijU}] [1,1] = [a_{ijL}, a_{ijU}] \left(\sum_{s \in \underline{n}} [a_{isL}, a_{isU}] \right) = \sum_{s \in \underline{n}} [a_{ijL} a_{isL}, a_{ijU} a_{isU}] \\ = \sum_{s \neq j} [a_{ijL} a_{isL}, a_{ijU} a_{isU}] + [a_{ijL}^2, a_{ijU}^2].$$

On the other hand, by (e), we have $\sum_{k \in \underline{n}} [a_{kjL} a_{ksL}, a_{kjU} a_{ksU}] = 0$ for $j \neq s$

i.e., $[a_{kjL} a_{ksL}, a_{kjU} a_{ksU}] = 0$ for $k \in \underline{n}$.

Therefore, $[a_{ijL}, a_{ijU}] = [a_{ijL}^2, a_{ijU}^2]$, which implies that $[a_{ijL}, a_{ijU}] \in I(L^1 n)$ and $A = [A_L, A_U] \in M(I(L^1 n))$.

By (i), f_A is injective mapping.

(k) $\Rightarrow$ (l), since $A = [A_L, A_U] \in M(I(L^1 n))$ and $I(L^1 n)$ is an interval distributive lattice by Lemma 4.1, it follows from Lemma 4.4, that A has an index.

(l) $\Rightarrow$ (g). Let $[A_L, A_U]$ has the index k and the period d , then $[A_L^{k+d}, A_U^{k+d}] = [A_L^k, A_U^k]$,

Since f_A is an injective mapping, therefore f_A^k is also an injective mapping.

Hence from $[A_L^k A_L^d, A_U^k A_U^d] = [A_L^{k+d}, A_U^{k+d}] = [A_L^k, A_U^k] = [A_L^k, A_U^k] [I_L, I_U] = A^k I$.

We have that $[A_L^d, A_U^d] = [I, I]$.

Example 4.1: Let us consider the Interval Incline square matrix defined as follows:

$$[a_{ijL}, a_{ijU}] = \begin{cases} [0.2, 0.5], & i = j \\ 0, & i \neq j \end{cases}.$$

Thus we have

$$\begin{pmatrix} [0.2, 0.5] & 0 & 0 & \dots & 0 \\ 0 & [0.2, 0.5] & 0 & \dots & 0 \\ 0 & 0 & [0.2, 0.5] & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & [0.2, 0.5] \end{pmatrix} \begin{pmatrix} [S_L, S_U] \\ [S_L, S_U] \\ [S_L, S_U] \\ \vdots \\ [S_L, S_U] \end{pmatrix} = \begin{pmatrix} [0.2S_L, 0.5S_U] \\ [0.2S_L, 0.5S_U] \\ [0.2S_L, 0.5S_U] \\ \vdots \\ [0.2S_L, 0.5S_U] \end{pmatrix}.$$

Hence the mapping f_A is surjective. However $[A_L A_L^T, A_U A_U^T] \neq [I, I]$ and so A is not invertible matrix.

Remark 4.2: Therefore we conclude that the IIM matrix A is invertible if and only if A is orthogonal.

Corollary 4.1: $([9])$ If $[A_L, A_U], [B_L, B_U] \in M(L^1 n)$ and $[A_L B_L, A_U B_U] = [I, I]$ then $[B_L, B_U] = [A_L^T, A_U^T]$ and $[B_L A_L, B_U A_U] = [I, I]$.

Proof: It follows from the Theorem 4.4

Corollary 4.2: $([17])$ If D is an interval distributive lattice and $A = [A_L, A_U] = [a_{ijL}, a_{ijU}] \in M(L^1 n)$ then the following statements are equivalent to each other.

- i. A is right invertible.
 - ii. A is left invertible
 - iii. A is invertible.
 - iv. The rows of A are orthogonal
- i.e., $\sum_{s \in \underline{n}} [a_{isL}, a_{isU}] = [1, 1]$ for all $i \in \underline{n}$ and $[a_{ikL} a_{jkL}, a_{ikU} a_{jkU}] = [0, 0]$ for $i \neq j$ and $k \in n$.
- i. The column of A are orthogonal
 - ii. The rows and columns of A are orthogonal respectively.
 - iii. $A^k = I$ for some positive integer k .
 - iv. f_A is bijective mapping.
 - v. f_A is surjective mapping.
 - vi. f_A is injective mapping
 - vii. f_A preserves the scalar product.

Proof: Since D is interval distributive lattice $I(L^1) = D$ and A has index, by Lemma 4.4. Hence the statements (l) coincide with (k) in Theorem 4.4. This completes the proof.

5. Conclusion:

We have extended the concept of incline to interval incline as a generalization of IVFS. Interval distributive lattice, etc. We obtained the basic properties and investigated the characterization of invertibility of IIMs.

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