



ON COMPUTING POWERS OF METALLIC RATIOS

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Abstract:

Among various amusing class of numbers in mathematics, metallic ratios occupy an important place in several explorations. The well known golden ratio, silver ratio and bronze ratios are special cases of metallic ratios. In this paper, we had obtained the integral power of n th Metallic ratio. As a special case, we obtain the integral power of golden ratio.

Key Words: Golden Ratio, Metallic Ratio, Pascal's Identity, Mathematical Induction, Floor Function

1. Introduction:

Numbers like golden ratio, silver ratio have fascinated many people throughout the globe for several centuries. These numbers can be generalized in to special class of numbers called "Metallic Ratios". In particular, the metallic ratio of order 1 is golden ratio, order 2 is silver ratio. These ratios have been used in architectures, paper sizes, paintings to name a few. The idea of studying metallic ratios is to have more general view of knowing how these numbers behave. In this paper, we had tried to obtain the integral power of the n th metallic ratio by proving three theorems. The answer turns out be an exciting expression.

2. Definitions:

2.1 Metallic Ratios:

The sequence of metallic ratios is defined through the recursive relation $M_{n+2} = nM_{n+1} + M_n$, $n \geq 0$ (2.1). The metallic ratio of order n , denoted by M_n is defined as the positive root of the auxiliary equation corresponding to (2.1), which is $x^2 - nx - 1 = 0$ (2.2).

$$\text{The roots of (2.2) are given by } M_n = \frac{n + \sqrt{n^2 + 4}}{2} \quad (2.3), \quad -\frac{1}{M_n} = \frac{n - \sqrt{n^2 + 4}}{2} \quad (2.4)$$

2.2 Special Cases:

If $n = 1$, then from (2.3), the number $M_1 = \frac{1 + \sqrt{5}}{2}$ (2.5) is called the Golden Ratio. Thus, the metallic ratio or order 1 is the golden ratio.

If $n = 2$, then from (2.3), the number $M_2 = \frac{2 + \sqrt{8}}{2} = 1 + \sqrt{2}$ (2.6) is called the Silver Ratio. Thus the metallic ratio of order 2 is the silver ratio.

If $n = 3$, then from (2.3), the number $M_3 = \frac{3 + \sqrt{13}}{2}$ (2.7) is called the Bronze Ratio. Thus the metallic ratio of order 3 is the bronze ratio.

To know more about other Metallic Ratios, see [1]

3.1 Fibonacci Numbers

The Fibonacci numbers are defined recursively by $F_{n+2} = F_{n+1} + F_n$, $n \geq 0$ (3.1) where $F_0 = 0, F_1 = 1$. Using this definition, the first few Fibonacci numbers are given by

$$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, F_8 = 21, F_9 = 34, F_{10} = 55, F_{11} = 89, F_{12} = 144, \dots$$

It is well known that the ratio of successive Fibonacci numbers tends to the golden ratio given in (2.5). For proof, see [1].

3.2 Pascal's Triangle:

The triangular array of numbers whose entries are binomial coefficients is called as Pascal's Triangle, named after French mathematician Blaise Pascal, though this triangle was known to earlier mathematicians. The Pascal's triangle with first nine rows is displayed below.

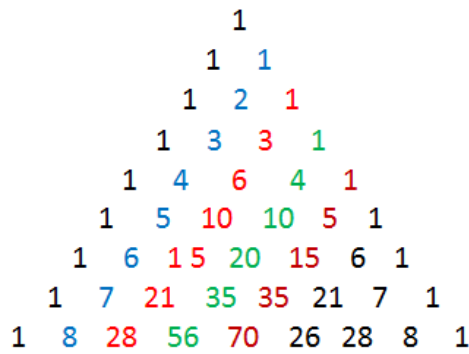
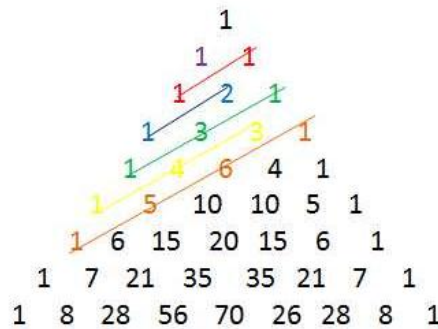


Figure 1: Pascal's Triangle

Pascal's triangle provides rich combinatorial properties that a whole book can be written to describe them. However, at present, we are primarily interested in exhibiting the wonderful connection between Fibonacci numbers and Pascal's triangle. This connection is visible from the figure shown below.



Fibonacci Sequence: 1, 1, 2, 3, 5, 8, 13

Figure 2: Generating Fibonacci Numbers from Pascal's Triangle

We note that for $n \geq 0, r \geq 0$ the r th entry in n th row of Pascal's triangle is the binomial coefficient

$$\binom{n}{r} = \frac{n!}{r! \times (n-r)!} \quad (3.2)$$

From Figure 2, we observe the following identities

$$F_2 = 1 = \binom{1}{0}, F_3 = 2 = \binom{2}{0} + \binom{1}{1}, F_4 = 3 = \binom{3}{0} + \binom{2}{1}, F_5 = 5 = \binom{4}{0} + \binom{3}{1} + \binom{2}{2},$$

$$F_6 = 8 = \binom{5}{0} + \binom{4}{1} + \binom{3}{2}, F_7 = 13 = \binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3}, \dots$$

The relationship between Fibonacci numbers and entries of Pascal's triangle is proved in the following theorem.

3.3 Theorem 1:

For any natural number n , the n th Fibonacci number is given by $F_n = \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-r}{r} \quad (3.3)$ where $\lfloor x \rfloor$ is the greatest integer $\leq x$, called the floor function of x .

Proof:

We will prove the desired result by mathematical induction. If $n = 1$, then we know that $F_1 = 1$. For this value of n , using (3.2) and (3.3), we find that $F_1 = \sum_{r=0}^0 \binom{1-1-r}{r} = \binom{0}{0} = 1$.

Thus the result is true for $n = 1$.

For $n = 2$, we know that $F_2 = 1$. For this value of n , using (3.2) and (3.3), we find that

$$F_2 = \sum_{r=0}^0 \binom{2-1-r}{r} = \binom{1}{0} = 1. \text{ We find that the result is also true for } n = 2.$$

Hence, by Induction Hypothesis, for any natural number n , we assume that the result is true for all values $1, 2, 3, \dots, n-1, n$ and show that the result is also true for $n+1$.

First we assume that n is odd. For such n , $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2} = \lfloor \frac{n-1}{2} \rfloor$ and $\lfloor \frac{n-2}{2} \rfloor = \frac{n-3}{2}$.

We know that the binomial coefficients satisfy the Pascal's identity

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \quad (3.4)$$

Hence, using (3.4), we have

$$\begin{aligned} \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-r}{r} &= \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-1-r}{r-1} + \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-1-r}{r} = \sum_{r=0}^{\frac{n-1}{2}} \binom{n-1-r}{r-1} + \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-r}{r} \\ &= \sum_{i=0}^{\frac{n-3}{2}} \binom{n-2-i}{i} + \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-r}{r} \\ &= \sum_{i=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-i}{i} + \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-r}{r} = F_{n-1} + F_n = F_{n+1} \end{aligned}$$

If we now assume that n is even, then for such n , $\lfloor \frac{n}{2} \rfloor = \frac{n}{2}$, $\lfloor \frac{n-1}{2} \rfloor + 1 = \lfloor \frac{n}{2} \rfloor$ and $\frac{n-2}{2} = \lfloor \frac{n-2}{2} \rfloor$. Moreover, we notice

that $n-1 - \lfloor \frac{n}{2} \rfloor < \lfloor \frac{n}{2} \rfloor$ since $n-1 - \lfloor \frac{n}{2} \rfloor = n-1 - \frac{n}{2} = \frac{n}{2} - 1 < \frac{n}{2} = \lfloor \frac{n}{2} \rfloor$.

Using these observations and Pascal's identity from (3.4), we have

$$\begin{aligned} \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-r}{r} &= \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-1-r}{r-1} + \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-1-r}{r} = \sum_{r=0}^{\frac{n}{2}} \binom{n-1-r}{r-1} + \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-1-r}{r} \\ &= \sum_{j=0}^{\frac{n-2}{2}} \binom{n-2-j}{j} + \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-r}{r} + \binom{n-1 - \lfloor \frac{n}{2} \rfloor}{\lfloor \frac{n}{2} \rfloor} \\ &= \sum_{j=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-2-j}{j} + \sum_{r=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-r}{r} + 0 = F_{n-1} + F_n = F_{n+1} \end{aligned}$$

Thus in either case, if the result is true up to n , then it also true for $n+1$. Hence by Induction Principle, equation (3.3) holds true for all natural numbers n . This completes the proof.

4. Integral Powers of Metallic Ratios:

4.1 In this section, we will obtain a closed expression for k th power of n th metallic ratio M_n . First, we try to compute first few integer powers of M_n .

From (2.3), we obtain $M_n^2 = nM_n + 1$ (4.1). Using this we can compute third, fourth, fifth and sixth powers of the metallic ratio M_n .

$$M_n^3 = (nM_n + 1) \times M_n = nM_n^2 + M_n = n(nM_n + 1) + M_n = (n^2 + 1)M_n + n \quad (4.2)$$

$$M_n^4 = [(n^2 + 1)M_n + n] \times M_n = (n^2 + 1)(nM_n + 1) + nM_n = (n^3 + 2n)M_n + (n^2 + 1) \quad (4.3)$$

$$\begin{aligned} M_n^5 &= [(n^3 + 2n)M_n + (n^2 + 1)] \times M_n = (n^3 + 2n)(nM_n + 1) + (n^2 + 1)M_n \\ &= (n^4 + 3n^2 + 1)M_n + (n^3 + 2n) \quad (4.4) \end{aligned}$$

$$\begin{aligned} M_n^6 &= [(n^4 + 3n^2 + 1)M_n + (n^3 + 2n)] \times M_n = (n^4 + 3n^2 + 1)(nM_n + 1) + (n^3 + 2n)M_n \\ &= (n^5 + 4n^3 + 3n)M_n + (n^4 + 3n^2 + 1) \quad (4.5) \end{aligned}$$

We can continue this procedure to compute any positive integer power of metallic ratio M_n .

4.2 Theorem 2:

If M_n is the metallic ratio of order n and if $k \geq 2$ is any positive integer then

$$M_n^k = \left[\sum_{r=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \binom{k-1-r}{r} n^{k-2r-1} \right] M_n + \left[\sum_{r=0}^{\left\lfloor \frac{k-2}{2} \right\rfloor} \binom{k-2-r}{r} n^{k-2r-2} \right] \quad (4.6)$$

Proof:

We will prove this by mathematical induction on k .

If $k = 2$, then from (4.1), we know that $M_n^2 = nM_n + 1$.

Substituting $k = 2$ in (4.6), we get

$$M_n^2 = \left[\sum_{r=0}^0 \binom{1-r}{r} n^{1-2r} \right] M_n + \left[\sum_{r=0}^0 \binom{-r}{r} n^{-2r} \right] = \binom{1}{0} n M_n + \binom{0}{0} = n M_n + 1.$$

Hence the result is true if $k = 2$. By Induction Hypothesis, assume that (4.6) is true for all values $2, 3, 4, \dots, k$. We will prove the result for $k + 1$. If we raise the metallic ratio M_n to the power $k + 1$, then we get

$$\begin{aligned} M_n^{k+1} &= M_n^k \times M_n = \left[\sum_{r=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \binom{k-1-r}{r} n^{k-2r-1} \right] M_n^2 + \left[\sum_{r=0}^{\left\lfloor \frac{k-2}{2} \right\rfloor} \binom{k-2-r}{r} n^{k-2r-2} \right] M_n \\ &= \left[\sum_{r=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \binom{k-1-r}{r} n^{k-2r-1} \right] (n M_n + 1) + \left[\sum_{r=0}^{\left\lfloor \frac{k-2}{2} \right\rfloor} \binom{k-2-r}{r} n^{k-2r-2} \right] M_n \\ &= \left[\sum_{r=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \binom{k-1-r}{r} n^{k-2r} + \sum_{r=0}^{\left\lfloor \frac{k-2}{2} \right\rfloor} \binom{k-2-r}{r} n^{k-2r-2} \right] M_n + \left[\sum_{r=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \binom{k-1-r}{r} n^{k-2r-1} \right] \\ &= \left[\sum_{r=0}^{\left\lfloor \frac{k}{2} \right\rfloor} \binom{k-r}{r} n^{k-2r} \right] M_n + \left[\sum_{r=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \binom{k-1-r}{r} n^{k-2r-1} \right] \end{aligned}$$

Thus the result is true for $k + 1$, if it is true up to k . Hence by Induction Principle, we find that (4.6) is true for all positive integers $k \geq 2$. This completes the proof.

4.3 Theorem 3:

If M_1 is the golden ratio (metallic ratio of order 1), then $M_1^k = F_k M_1 + F_{k-1}$ (4.7) where F_k is the k th Fibonacci number.

Proof:

First, we consider $n = 1$ in (4.6) to obtain

$$M_1^k = \left[\sum_{r=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \binom{k-1-r}{r} \right] M_1 + \left[\sum_{r=0}^{\left\lfloor \frac{k-2}{2} \right\rfloor} \binom{k-2-r}{r} \right].$$

Now using (3.3), we get $M_1^k = F_k M_1 + F_{k-1}$. This completes the proof.

5. Conclusion:

The connection between Fibonacci numbers and entries of Pascal's triangle is established in Theorem 1 through equation (3.3). Though the relationship is widely known, the proof for the same is not so well known and it is dealt clearly in this paper. The primary purpose of the paper, namely determining k th power of the n th metallic ratio M_n is achieved through a nice closed expression in equation (4.6) of theorem 2. One can notice that by substituting values of $k = 2, 3, 4, 5, 6$ in (4.6), we obtain equations (4.1) to (4.6) respectively. Thus equation (4.6) is the general formula for finding integral powers of metallic ratios. This result is very new and done for the first time in this paper.

Using the more general result, we had obtained the integral powers of golden ratio, the metallic ratio of order 1, in terms of Fibonacci numbers. Though this result is well known in the literature, it is obtained as a corollary of the more general result established in equation (4.6). Moreover, from (4.6), we can easily compute desired integer power of any metallic ratio. Curiously, in establishing the more general result, we saw that the coefficients in equation (4.6) are Fibonacci numbers. Thus Fibonacci numbers occur not just in powers of golden ratio, but it does so for general metallic ratios as well. This surprising consequence was revealed in this paper.

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